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NOTE

THE *i*th RAMSEY NUMBER FOR MATCHINGS

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The ith Ramsey number for matchings is determined. In addition, our results lead to the calculation of the Ramsey index for matchings.

The purpose of this paper is to calculate the ith Ramsey number for matchings. In order to state our results, we will need some notation. Any undefined notation follows Behzad, Chartrand and Lesniak-Foster [1]. For $i, n \in Z^+$, we write $K(i; n)$ instead of $K(i, \ldots, i)$ to denote the complete n-partite graph whose parts each contain i vertices. In particular, $K(1, n)$ is the usual star graph while $K(1; n)$ is the complete graph K_n. For $k \in Z^+$ and graphs $G, G_1, \ldots, G_k$, we write $G \to (G_1, \ldots, G_k)$ to mean that if $G = F_1 \oplus \cdots \oplus F_k$ is an arbitrary (edge) factorization of G, then $G_j \subset F_j$ for some j.

For $i, k \in Z^+$ and graphs $G_1, \ldots, G_k$, the ith *Ramsey number* $r_i(G_1, \ldots, G_k)$ is defined to be the least positive integer p such that $K(i; p) \to (G_1, \ldots, G_k)$. In particular, $r_1(G_1, \ldots, G_k)$ is the generalized Ramsey number $r(G_1, \ldots, G_k)$. Moreover, if $i \leqslant j$ and $K(i; p) \to (G_1, \ldots, G_k)$, then $K(i; p) \subset K(j; p)$ and therefore $K(j; p) \to (G_1, \ldots, G_k)$. So $r_i(G_1, \ldots, G_k) \geqslant r_j(G_1, \ldots, G_k)$ whenever $i \leqslant j$ and therefore $r_i(G_1, \ldots, G_k) \leqslant r(G_1, \ldots, G_k)$ for each $i \in Z^+$. It follows that ith Ramsey numbers do indeed exist. In addition, the sequence $\{r_i(G_1, \ldots, G_k)\}_{i=1}^{\infty}$ is a non-increasing sequence of positive integers and therefore convergent. The *Ramsey index* $i(G_1, \ldots, G_k)$ is defined to be the least positive integer I such that $r_I(G_1, \ldots, G_k) = \lim_{i \to \infty} r_i(G_1, \ldots, G_k)$. Obviously, $r_i(K_2, \ldots, K_2) = 2$ for each i and therefore $i(K_2, \ldots, K_2) = 1$.

The above “Ramsey numbers” were introduced by Benedict [2]. In addition, Benedict extended the work of Burr and Roberts [5] by calculating the ith Ramsey number and Ramsey index for stars. In a further attempt to extend currently known ith Ramsey numbers, we calculate the ith Ramsey number and Ramsey index for matchings. We require the following theorem.

Theorem 1 (Cockayne and Lorimer [6]). *If $k \in Z^+$ and $m_1, \ldots, m_k \in Z^+$ with $m_1 \geqslant m_j$ for each j, then $r(m_1K_2, \ldots, m_kK_2) = m_1 + 1 + \sum_{j=1}^{k} (m_j - 1)$.*

Now we present some preliminary results before establishing the main theorems. The symbol $[x]$, for a real number x, denotes the greatest integer less than or equal to x.

Lemma 2. *For i, $k \in Z^+$ and graphs $G_1, \ldots, G_k$, $r_i(G_1, \ldots, G_k) \geq [(r(G_1, \ldots, G_k) - 1)/i] + 1$.*

Proof. Since $[(r(G_1, \ldots, G_k) - 1)/i]i \leq r(G_1, \ldots, G_k) - 1$, it follows that $K(i; [(r(G_1, \ldots, G_k) - 1)/i]) \not\rightarrow (G_1, \ldots, G_k)$. Hence $r_i(G_1, \ldots, G_k) \geq [(r(G_1, \ldots, G_k) - 1)/i] + 1$. □

Corollary 3. *Let $i, k \in Z^+$ and let $m_1, \ldots, m_k \in Z^+$ with $m_1 \geq m_j$ for each j. If $s = \sum_{j=1}^{k} (m_j - 1)$, then $K(i; [(m_1 + s)/i]) \not\rightarrow (m_1K_2, \ldots, m_kK_2)$.*

Proof. Since $r(m_1K_2, \ldots, m_kK_2) = m_1 + 1 + s$ by Theorem 1, it follows that $r_i(m_1K_2, \ldots, m_kK_2) \geq [(r(m_1K_2, \ldots, m_kK_2) - 1)/i] + 1 = [(m_1 + s)/i] + 1$ by Lemma 2. Hence $K(i; [(m_1 + s)/i]) \not\rightarrow (m_1K_2, \ldots, m_kK_2)$. □

Lemma 4. *Let i, k, $x \in Z^+$ and let $m_1, \ldots, m_k \in Z^+$. Let $s = \sum_{j=1}^{k} (m_j - 1)$. If G is a graph with $s + x$ vertices and an independent set of at least x vertices, then $G \not\rightarrow (m_1K_2, \ldots, m_kK_2)$.*

Proof. Partition the vertices of G into the sets $V_0, V_1, \ldots, V_k$, where V_0 contains x independent vertices and V_j contains $m_j - 1$ vertices for each $j \neq 0$. Define the factorization $G = F_1 \oplus \cdots \oplus F_k$ such that F_j $(1 \leq j \leq k)$ contains all edges having one endpoint in V_j and the other in some V_t with $t \leq j$. Then $m_jK_2 \not\subseteq F_j$ for each j and therefore $G \not\rightarrow (m_1K_2, \ldots, m_kK_2)$. □

Corollary 5. *Let $i, k \in Z^+$ and let $m_1, \ldots, m_k \in Z^+$ with $m_1 \geq m_j$ for each j. Let $s = \sum_{j=1}^{k} (m_j - 1)$ and write $m_1 + s = qi + r$ for integers q and r with $0 \leq r < i$. If $r \geq m_1$, then $K(i; [(m_1 + s)/i] + 1) \not\rightarrow (m_1K_2, \ldots, m_kK_2)$.*

Proof. The corollary follows immediately from Lemma 4. □

Lemma 6. *Let $i, k \in Z^+$ and let $m_1, \ldots, m_k \in Z^+$. Let $s = \sum_{j=1}^{k} (m_j - 1)$. If G is a graph with at least $r(m_1K_2, \ldots, m_kK_2)$ vertices and* $\deg(u) + \deg(v) > 2s$ *for every pair of nonadjacent vertices u and v, then $G \rightarrow (m_1K_2, \ldots, m_kK_2)$.*

Proof. Assume that $G \not\rightarrow (m_1K_2, \ldots, m_kK_2)$. If we successively join pairs of nonadjacent vertices of G, the resulting graph will eventually arrow $(m_1K_2, \ldots, m_kK_2)$ since G has at least $r(m_1K_2, \ldots, m_kK_2)$ vertices. Let $e_1, \ldots, e_t$ be a minimal sequence of edges added to G in the sense that if H is the resulting graph, then $H \rightarrow (m_1K_2, \ldots, m_kK_2)$ but $H - e_t \not\rightarrow (m_1K_2, \ldots, m_kK_2)$. By our assumption that $G \not\rightarrow (m_1K_2, \ldots, m_kK_2)$, it follows that $t \geq 1$.

Let $H - e_t = F_1 \oplus \cdots \oplus F_k$ be a factorization such that $m_j K_2 \not\subset F_j$ for each j. Consider the factorization $H = (F_1 + e_t) \oplus F_2 \oplus \cdots \oplus F_k$. Clearly, $m_1 K_2 \subset F_1 + e_t$ and, moreover, e_t must be an edge in any such $m_1 K_2$. Let $e_t = uv$ for vertices u and v, and let $x_1 y_1, \ldots, x_{m_1-1} y_{m_1-1}$ be the remaining edges of some $m_1 K_2$ in $F_1 + e_t$. Since $m_1 K_2 \not\subset F_1$, each vertex of F_1 adjacent to u or v comes from $x_1, \ldots, x_{m_1-1}, y_1, \ldots, y_{m_1-1}$. In addition, at most two edges in F_1 can have one endpoint in the set $\{u, v\}$ and the other in $\{x_j, y_j\}$ for some j. Hence $\deg_{F_1}(u) + \deg_{F_1}(v) \leqslant 2(m_1 - 1)$.

Similarly, the factorization $H = F_1 \oplus \cdots \oplus F_{j-1} \oplus (F_j + e_t) \oplus F_{j+1} \oplus \cdots \oplus F_k$ implies that $\deg_{F_j}(u) + \deg_{F_j}(v) \leqslant 2(m_j - 1)$ for each j. Thus $\deg_{H-e_t}(u) + \deg_{H-e_t}(v) \leqslant 2s$. But $G \subset H - e_t$ and therefore $\deg_{H-e_t}(u) + \deg_{H-e_t}(v) > 2s$, a contradiction. Hence $G \rightarrow (m_1 K_2, \ldots, m_k K_2)$. □

Corollary 7. *Let $i, k \in Z^+$ and let $m_1, \ldots, m_k \in Z^+$ with $m_1 \geqslant m_j$ for each j. Let $s = \sum_{j=1}^{k} (m_j - 1)$ and write $m_1 + s = qi + r$ for integers q and r with $0 \leqslant r < i$.*

(i) *If $r < m_1$, then $K(i; [(m_1 + s)/i] + 1) \rightarrow (m_1 K_2, \ldots, m_k K_2)$.*

(ii) *If $r \geqslant m_1$, then $K(i; [(m_1 + s)/i] + 2) \rightarrow (m_1 K_2, \ldots, m_k K_2)$.*

Proof. The corollary follows immediately from Lemma 6. □

We are now prepared to present the main results of the paper.

Theorem 8. *Let $i, k \in Z^+$ and let $m_1, \ldots, m_k \in Z^+$ with $m_1 \geqslant m_j$ for each j. Let $s = \sum_{j=1}^{k} (m_j - 1)$ and write $m_1 + s = qi + r$ for integers q and r with $0 \leqslant r < i$. Then $r_i(m_1 K_2, \ldots, m_k K_2) = [(m_1 + s)/i] + \theta$ with $\theta = 1$ if $r < m_1$ and $\theta = 2$ if $r \geqslant m_1$.*

Proof. The theorem follows immediately from Corollaries 3, 5 and 7. □

Theorem 9. *Let $k \in Z^+$ and let $m_1, \ldots, m_k \in Z^+$. Then*

$$i(m_1 K_2, \ldots, m_k K_2) = \sum_{j=1}^{k} (m_j - 1) + 1.$$

Proof. Let $s = \sum_{j=1}^{k} (m_j - 1)$. If $s = 0$, then $m_j = 1$ for each j and therefore $i(m_1 K_2, \ldots, m_k K_2) = i(K_2, \ldots, K_2) = 1$. So we can assume that $s \neq 0$. Clearly $\lim_{i \to \infty} r_i(m_1 K_2, \ldots, m_k K_2) = 2$, so we want to show that $r_s(m_1 K_2, \ldots, m_k K_2) = 3$ while $r_{s+1}(m_1 K_2, \ldots, m_k K_2) = 2$.

We can assume that $m_1 \geqslant m_j$ for each j and therefore $m_1 > 1$. In addition, $s \geqslant m_1 - 1$, so either $s = m_1 - 1$, $s = m_1$ or $s > m_1$. If $s = m_1 - 1$, then $m_1 + s = 2s + 1$ and therefore $r_s(m_1 K_2, \ldots, m_k K_2) = 2 + 1 = 3$ by Theorem 8. If $s = m_1$, then $m_1 + s = 2s$ and therefore $r_s(m_1 K_2, \ldots, m_k K_2) = 2 + 1 = 3$ by Theorem 8. If $s > m_1$, then $m_1 + s = 1s + m_1$ and therefore $r_s(m_1 K_2, \ldots, m_k K_2) = 1 + 2 = 3$ by Theorem 8. Hence $r_s(m_1 K_2, \ldots, m_k K_2) = 3$.

Similarly, since $s+1 \geqslant m_1 > m_1 - 1$, it follows that $m_1 + s = 1(s+1) + (m_1 - 1)$ and therefore $r_{s+1}(m_1K_2, \ldots, m_kK_2) = 1 + 1 = 2$ by Theorem 8. It follows that $i(m_1K_2, \ldots, m_kK_2) = s + 1$. □

Although the ith Ramsey numbers for matchings may be of doubtful interest, we believe that the results of this article may be applicable in other areas of Ramsey theory. In particular, as noted by the Referee, Lemmas 4 and 6 may be relevant in the study of Ramsey-minimal graphs for matchings [3]. In addition, a more direct study of Ramsey indices may prove worthwhile in the areas of chromatic Ramsey numbers [4] and bipartite Ramsey numbers [7].

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